

Statistical signature of vortex filaments: dog or tail? Talk given at QFS 2016.

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Abstract The title of the paper coincides with the title of a paragraph in the famous book by U. Frisch (1995) on classical turbulence. In this paragraph the author discussed the role of statistical dynamics of vortex filaments in the theory of turbulence and put the above question. In other words, whether the main properties of turbulence (cascade, scaling laws) are the sequence of the vortex line dynamics or the latter have only marginal signature. Quantum fluids, where the vortex filaments are the real objects, give an excellent opportunity to explore the role of discrete vortices in turbulent phenomena. The aim of this paper is to discuss which elements of vortex dynamics would lead to the main ingredients of the theory of turbulence. We discuss how the nonlinear dynamics of vortex filaments can result in an exchange of energy between different scales, the formation of the Kolmogorov-type energy spectra and the decay of turbulence.

Keywords superfluidity, vortices, quantum turbulence

1 Introduction.

The idea that classical turbulence can be modeled by a set of slim vortex tubes (or vortex sheets) has been discussed for quite a long time (See [1]). In classical fluids, the concept of thin vortex tubes is a rather fruitful mathematical model. Quantum fluids, where the vortex filaments are real objects, give an excellent opportunity for the study of the question, whether the dynamics of a set of vortex lines is able to reproduce the properties of real hydrodynamic turbulence. The main goal of this paper is to discuss which elements of vortex dynamics would lead to main ingredients of theory of turbulence.

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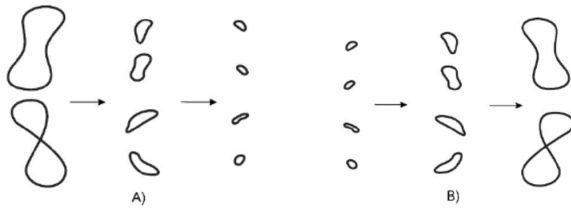


Fig. 1 (Color online) The cascade like breaking down and merging of Gaussian vortex loop due to reconnection and self-reconnection processes.

In the paper we restrict ourselves to the most manifest features of the turbulent flow, such as the exchange of energy between the different scales and the Kolmogorov-type energy spectra. We discuss the exchange of energy due to recombinations (splitting and fusion) of the vortex loops and due to the nonlinear stochastic deformation of initially smooth ring. We also describe the attempts to obtain the Kolmogorov type spectra from the configurations of vortex lines, which appeared as a result of the tangle evolution. In particular, we investigate the reconnecting lines and inhomogeneous vortex bundles. We also consider the problem of decay of turbulence and its relation to the such phenomena as the KW cascade and the emission of small loops. And finally we intend to summarize inquiring, whether outlined issues are relevant to classical turbulence, or that are other phenomena.

2 Exchange of energy between different scales. Vortex loops kinetics

Recombination of Gaussian loops. This model assumes that the vortex tangle consists of a set of many closed vortex loops, which undergo an enormous number of reconnections and self-reconnections, reaching (for typical experiments) values of order of several millions collisions per second (per cm^3). Thus, in the full statement of the problem we have to deal with a set of objects (vortex loops), which do not have a fixed number of elements, they can be born and die. One of the possible treatments of this problem is to impose the vortex loops to have the random walking structure. The idea that the random one-dimensional topological defects have a random walking structure is widespread (see, e.g., books by Kleinert [2]). For vortex loops in superfluid helium this idea is realized in form of the so-called generalized Wiener distribution, which takes into account the anisotropy, polarization and the finite curvature. It has been developed in [3]. That approach allows to study kinetics of vortex loops. The corresponding study was performed in works [4, 5, 6]. To demonstrate the existence of energy cascade in space of scales it was chosen the usual variant of the Wiener distribution with an elementary step ξ_0 of the order of intervortex distance $\xi_0 \sim \delta = \mathcal{L}^{-1/2}$, where \mathcal{L} is the vortex line density (VLD).

Then, the only degree of freedom which exists is the length l of the loop, and all interactions reduce to collisions and recombinations (see Fig. 1). The corresponding problem can be studied on the basis of the Boltzmann-type “kinetic equation” and has an exact, power-like solution for the distribution function of number density of loops $n(l)$ with length l , namely $n(l) = \text{const} * l^{-5/2}$. This theory gives a series of predictions which can be associated with quasi-classical behavior of quantum turbulence.

The constant flux of the energy. The found solution corresponds to a nonequilibrium state, it describes a flux of length density $L(l, t) = ln(l, t)$, which is length, accumulated in loops of size l in l -space. The term “flux” here means just the redistribution of length among the loops due to reconnections. Conservation of the vortex line density can be expressed in the form of a continuity equation for the length density $L(l, t)$

$$\frac{\partial L(l, t)}{\partial t} + \frac{\partial P(l)}{\partial l} = 0. \quad (1)$$

In the local induction approximation (LIA) the energy of a line is proportional to its length, $E \propto \kappa^2 L$, therefore Eq.1 describes the constant flux of the vortex energy $P^E(l) = \text{const}$ in space of scales (which are just the loop sizes l). The flux P^E consists of two contributions. The first, positive one, is related to merging of loops, and responsible for delivering the energy into large scales. The second, negative contribution appears due to breaking-down of loops (see Fig. 1) describes flux of energy to small scales. Depending on the temperature either one or the other can prevail, it depends just on the temperature behavior of structure constants of the vortex tangle. In particular, for $T = 0$, energy flux $P^E < 0$, the energy is transferred into the region of small scales. This corresponds to the direct cascade in classical turbulence.

Effective Kinematic Viscosity. We can rewrite the flux of energy using VLD \mathcal{L} (up to factor $(1/4\pi) \ln(1/\mathcal{L}^{1/2} a_0)$, a_0 - the vortex core size)

$$P^E = C_F \kappa (\kappa \mathcal{L})^2. \quad (2)$$

We named this constant C_F in honor of Feynman, who was the first who discussed the decay of the superfluid turbulence due to the cascade-like breaking down of vortex loops. Expression of type (2) has the widely accepted form (see. e.g., [7]). In case of the counterflowing turbulence it appeared directly from the Vinen equation. In the case of the quasi-classical turbulence, this form of (2) is a consequence of the fact that the dissipation rate is proportional to the squared averaged vorticity $|\omega|^2$, and of the supposition that $\langle |\omega| \rangle = \kappa \mathcal{L}$.

We obtained relation (2) on a different ground. Referring for details to works [4,5,6] we can estimate the coefficient in (2) as $C_F \approx -0.25$. This result agrees by the order with the value of the effective kinematic viscosity ν' , extracted from experiments on decay of vortex tangle.

Resuming, it can be argued that the process of breaking down of vortex loops can be associated with the main feature of turbulence – the constant flux of energy in space of scales, and the relation between this flux and VLD (2) agrees with experimental data.

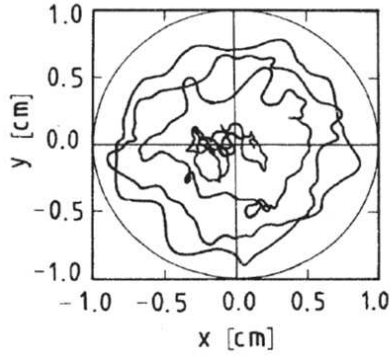


Fig. 2 (Color online) Evolution of vortex ring under influence of external random force in local approach (Eq. 3). As predicted, an consequent arising of higher harmonics takes place leading eventually to an entanglement of the initially smooth vortex loop.

3 Exchange of energy between different scales. Stochastic deformation of loops

Another mechanism of exchange of the energy between scales, inherent for turbulent phenomena, is related to nonlinear dynamics of a single vortex filament (ignoring the reconnections.). Here I describe the solution of the problem of chaotic distortions of a vortex loop [8,9]. In the LIA the chaotic motion of a quantized vortex filament obeys the Langevin type equation

$$\frac{d\mathbf{s}(\xi, t)}{dt} = \beta \mathbf{s}' \times \mathbf{s}'' + \eta(\xi, t) + \zeta(\xi, t). \quad (3)$$

Here $\mathbf{s}(\xi, t)$ is the position vector of line points labelled by the arc length ξ , which varies in interval $0 < \xi < 2\pi$, the quantity $\eta(\xi, t)$ stands for dissipation, which acts for marginally small scales. Quantities \mathbf{s}' and \mathbf{s}'' are the first and second derivatives of the function $\mathbf{s}(\xi, t)$ with respect to variable ξ . The external Langevin force $\zeta(\xi, t)$ is supposed to be Gaussian with a correlator concentrated on large scales. An example of evolution of an initially smooth vortex ring, obeying evolution equation (3), is depicted in Fig. 2.

One of key quantities in statistical theories is the spectral density tensor $S_p^{\alpha\beta}$, defined as $S_p^{\alpha\beta} \delta(p + p_1) = \langle \mathbf{s}_p^\alpha \mathbf{s}_{p_1}^\beta \rangle$ (p is the one-dimensional wave vector, conjugated to variable ξ). It is responsible for the correlation between tangent vectors, and for distribution of energy in the space of scale, etc. The quantity $S_p^{\alpha\beta}$ had been calculated in works [8,9]. The problem has the analytical solution based on a very popular theoretical trick – the so called Direct Interaction Approximation (DIA) for diagramming technique elaborated in classical turbulence by Wyld [10]. According calculations leads to the result that the spectral density tensor $S_p^{\alpha\beta}$ is a power function of the wave number

$$S_p^{\alpha\beta} = Cp^{-5}, \quad (4)$$

Another important result concerns the energy conservation in the p space

$$\frac{\partial E_p}{\partial t} + \frac{\partial P_p}{\partial p} = I_+(p) - I_-(p) \quad (5)$$

where $E_p = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \mathbf{s}'_p \mathbf{s}'_{-p} e^{-ip\xi} d\xi$ is the length (energy in LIA) and P_p^E is the flux of this quantity in Fourier space (or, equally, in space of scales). The right-hand side of Eq. (5) describes the creation of additional curvature at a rate of $I_+^K(p)$ due to external force and the annihilation of it due to the dissipative mechanism (at a rate $-I_-^K(p)$). In a region of wave numbers p remote from both region of pumping p_+ and sink p_- , $p_+ \ll p \ll p_-$, the so called inertial interval, derivative $\partial P_p^E / \partial p = 0$, so P_p^E is constant equal, say, P^E .

Again, the problem stated above, describes one of the main features of turbulence – the constant flux of energy in space of scales.

4 The Kolmogorov type spectra generated by vortex filaments

Let us discuss how the dynamics of vortex filaments can result in the Kolmogorov-type spectrum. The energy E of vortex line configuration $\{\mathbf{s}'(\xi)\}$ is the integral over all wave vectors \mathbf{k} (see [11]),

$$E = \left\langle \frac{\rho_s \kappa^2}{2(2\pi)^3} \int_{\mathbf{k}} \frac{d^3 \mathbf{k}}{k^2} \int_0^L \int_0^L \mathbf{s}'(\xi_1) \cdot \mathbf{s}'(\xi_2) d\xi_1 d\xi_2 \exp \left[i \int_{\xi_1}^{\xi_2} \mathbf{k} \cdot \mathbf{s}'(\tilde{\xi}) d\tilde{\xi} \right] \right\rangle. \quad (6)$$

The brackets $\langle \cdot \rangle$ imply an averaging over the configurations of vortex loops. I will describe two cases of configurations $\{\mathbf{s}'(\xi)\}$, which indeed lead to the Kolmogorov spectrum. They are the reconnecting filaments, the collapsing nonuniform vortex bundle, which appears as a result of vortex dynamics (See. Fig. 3 and Fig.4). The spectra for these configurations were calculated with the use of formula (6) in works by the author [12,13]. From Fig. 3 it is seen that for reconnecting lines the spectrum is close to the $E(k) \propto k^{-5/3}$. The interval of wave numbers where the spectrum $E(k) \approx k^{-5/3}$ (straight line) is observed, is regulated by the curvature of the kink and intervortex space δ , which was chosen to be equal to unity. In reality this spectrum covers a maximum about 1.5 decades around $k \approx 2\pi/\delta$. It should be stressed, however, that in the key numerical works (see references in [12]), the ranges for wave number are also of the order of one decade around $k \approx 2\pi/\delta$.

As for the collapsing nonuniform bundle, then the spectrum $E(k)$ scales as $1/k^{1+4/\lambda}$, where λ is the increment of nonuniformity of the vortex lattice on the plane transverse to the bundle (see [14]). Supposing that the bundle evolves similarly to the vorticity field in classic hydrodynamics (See e.g., [15]), then $\lambda = 6$ and the spectrum $E(k) \propto k^{-5/3}$.

The idea that collapsing singular solutions can play a significant role in formation of turbulent spectra is being intensively discussed now (see e.g. [15],

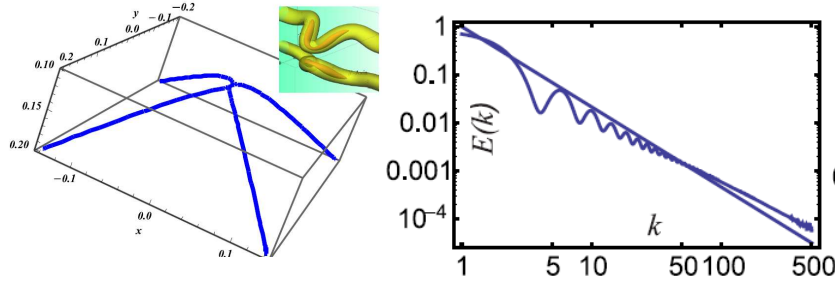


Fig. 3 (Color online) Left. The touching quasi-hyperbolae describing the collapsing lines. In the inset we set (as an example) the kinks on the anti-parallel collapsing vortex tubes obtained in numerical simulation. Right. The energy spectrum induced by this configuration.

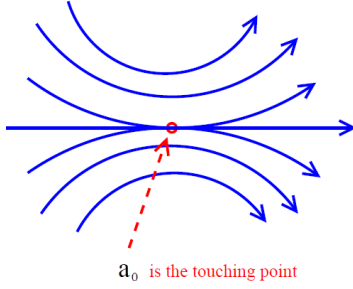


Fig. 4 (Color online) Schematic picture illustrating the vortex bundle collapse [16]. The initially regular distribution of vorticity spontaneously concentrates, collapsing in some point \mathbf{a}_0 and forming the singular structure.

[17]). The classical examples of such type spectra are the Phillips spectrum for water-wind waves, created by white caps – wedges on water surface.

One more interesting observation is related to the previous paragraph. From formula 6 it follows that for the Gaussian loop with the power-like correlator $\langle \mathbf{s}'(\xi_1) \cdot \mathbf{s}'(\xi_2) \rangle \propto (\xi_1 - \xi_2)^\lambda$, the energy spectrum is also a power-like function (see [11])

$$E(k) \propto k^{-\frac{2\lambda+2}{\lambda-2}}. \quad (7)$$

Let's apply this consideration to the stochastically deformed vortex loop described in the previous subsection (See Eq. 4). The one dimensional spectrum along the line $\langle \mathbf{s}_p \mathbf{s}_{-p} \rangle \propto p^{-5}$ implies that the correlation function $\langle \mathbf{s}'(\xi_1) \cdot \mathbf{s}'(\xi_2) \rangle$ should scale as $(\xi_1 - \xi_2)^4$ (See [1]). Applying this to [1] leads to the Kolmogorov spectrum $E(k) \propto k^{-5/3}$. This result, however, should be taken with caution. Indeed, the chaotic loop, discussed above is not the Gaussian one, and knowledge of the correlation function of second order is not enough to apply relation (7) directly.

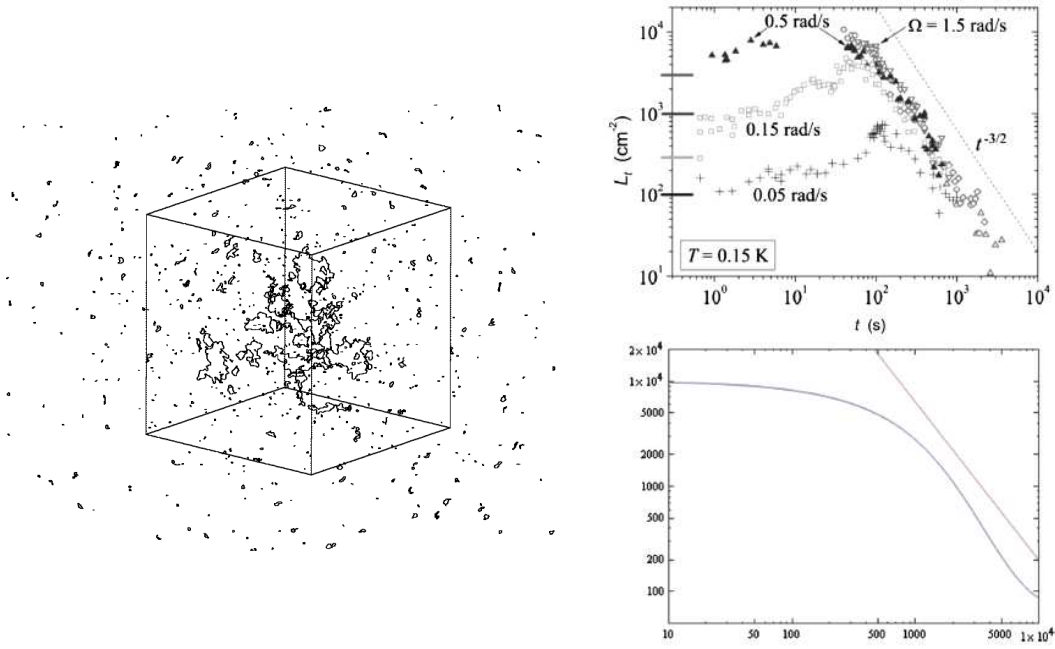


Fig. 5 (Color online) Left. The emission of small loops from the bulk. Results of direct numerical simulation (see, [18]). Right. Comparison with experiment. In the upper picture there is shown the temporal attenuation of the vortex line density, obtained on experiment [19]. Temporal behavior of same quantity, calculated from theoretical consideration [20].

5 Free decay of quantum turbulence.

We submitted several arguments that the main features of classical turbulence - energy cascade and the Kolmogorov like spectra may appear as a result of the dynamics of vortex filament. However, one intriguing question is left. In classical turbulence energy dissipates at the end of inertial interval due to viscosity. In superfluids viscosity is zero, and the question arises how the energy, transferred into the region of small scales, disappears? Currently, there are several views on the solution to this paradox.

The first one, initially proposed by Svistunov is connected to a cascade of nonlinear Kelvin waves. Due to nonlinearity, very small distortions, which move with very large velocities appear in the system (cf. with 3). As it is known from classical hydrodynamics, vortices moving with a velocity V radiate sound, and intensity I of the radiated sound is proportional to the Mach number in fifth power $I \propto (V/c)^5$ (c is the sound velocity). It implies that in order to obtain the significant effect, the quantized vortex line should have very large curvature or be placed very near the neighboring vortex or near the boundary.

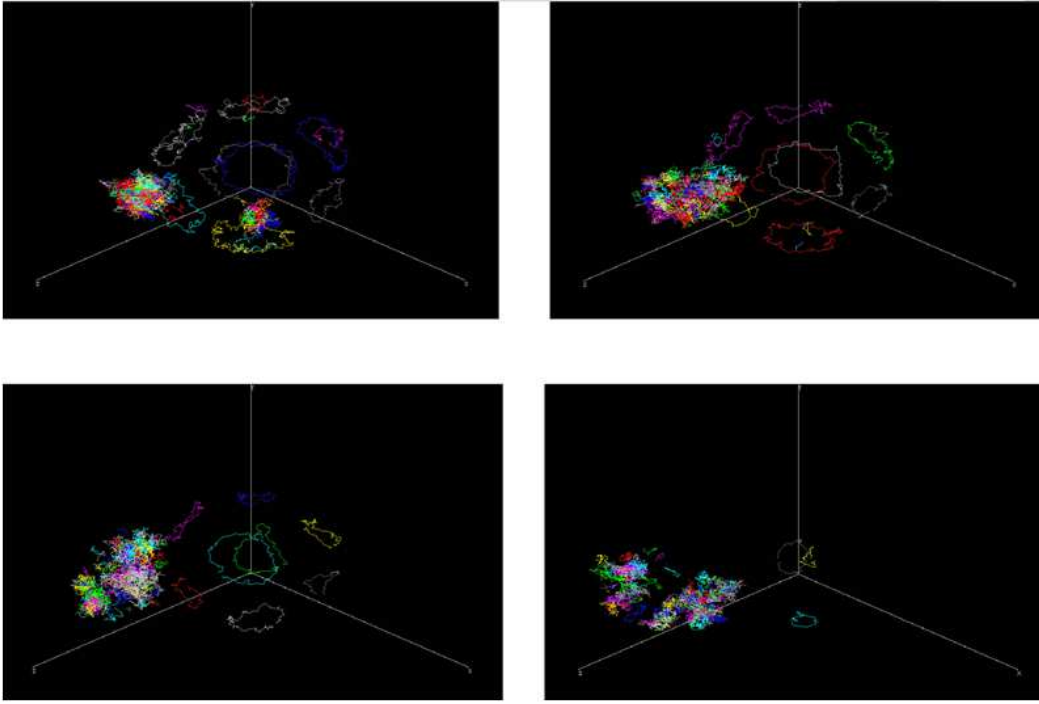


Fig. 6 (Color online) Schematic picture illustrating the vortex bundle collapse [16]. The regular initially distribution of vorticity spontaneously concentrates, collapsing in some point \mathbf{a}_0 and forming the singular structure.

Another mechanism is related to the emission of small loops from the bulk. As discussed earlier, during the reconnection cascade, there appear large number of very small loops. But these loops, because of their small size have large mobility and escape from the volume. This process occurs in the diffusion-like manner, the according calculations agree with experimental data.

6 Intermittency

An interesting phenomena, reminiscent of intermittency, was revealed in work by Kondaurova et al. [21]. The vortex tangle does not have an uniform structure, but, on the contrary consists of various clusters with high vorticity, which spontaneously appear and disappear in different places of space.

7 Conclusion and Discussion

Thus, we demonstrated that some results obtained in quantum fluids, which were considered as arguments in favour of the idea of modeling classical tur-

bulence with a set of chaotic quantized vortices, can be explained in the frame of theoretical models dealing with dynamics of vortex filaments.

The format of the paper, which reflects the presentation at QFS 2016, and also the page limitation did not allow to cover many other results on this topic, therefore I restrict myself mainly by my previous results. Among other theoretical works dealing with dynamics of chaotic vortices I would like to mention the activity on stochastic nonlinear Kelvin waves and their role in forming an energy cascade, and dissipation. This topic was elaborated in works by Vinen [22], Kozik and Svistunov [23] and in papers by L'vov and Nazarenko with coauthors [24]. Migdal [25] developed the functional formalism for the study of chaotic vortex filaments with possible application for the problem of turbulence. The role of chaotic superfluid vortex lines in a model of turbulent flow was also discussed in the work by Barenghi with coauthors [26]. In work by Kondaurova [18] it was demonstrated how the kinetics of loops resulted in the quasi-classical decay of quantum turbulence.

This paper should be regarded as an illustration to the idea that the main properties of turbulence (cascade, scaling laws) are the sequence of the vortex line dynamics. Now, there appears the following bifurcation. An optimistic view is that these approaches do relate to the real turbulence and we have the new vision on processes occurring in turbulent flow. A pessimistic view is that the phenomena described above have nothing to do with the real turbulence and all coincidences are occasional.

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